LINEAR ALGEBRA (finite-dimensional vector spaces)

for A, B $n \times n$ over $\mathbb{F}$, A similar to B if $\exists$ $n \times n$ P s.t. $B = P^{-1}AP$.

(↳ same det, rank, trace, evalues, char poly.)

- for $T: V \to V$ and $B = \{v_1, \dots, v_n\}$ basis of V:

$$\left.\begin{aligned} T(v_1) &= a_{11}v_1 + \dots + a_{n1}v_n \\ &\vdots \\ T(v_n) &= a_{1n}v_1 + \dots + a_{nn}v_n \end{aligned}\right\} \quad [T]_B = (a_{ij}) = \begin{array}{c} \begin{array}{cc} T(v_1) & T(v_n) \\ \downarrow & \downarrow \end{array} \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{array}$$

ALGEBRAIC MULTIPLICITY: $T: V \to V$ with char poly $c_T(x)$, then

$p(x) = (x-\lambda)^{a(\lambda)} q(x)$, where λ is not root of $q(x)$. $a(\lambda)$ algebraic mult.

GEOMETRIC MULTIPLICITY: $g(\lambda) = \dim E_\lambda = \dim(\ker(T - \lambda I))$.

- $g(\lambda) \le a(\lambda)$ and equality iff T diagonalisable.

DIRECT SUM: $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$ for subspaces V_i of V, if all $v \in V$ can be expressed as $v = v_1 + \dots + v_k$ for unique $v_i \in V$.

↳ e.g. $\mathbb{R}^2 = Sp\begin{pmatrix}1\\0\end{pmatrix} \oplus Sp\begin{pmatrix}0\\1\end{pmatrix}$.

PROP: $V = V_1 \oplus \dots \oplus V_k \iff \dim V = \sum_{i=1}^{k} \dim V_i$ and if B_i basis for V_i then $B_1 \cup \dots \cup B_k$ basis for V.

T-INVARIANT: $T: V \to V$, $W \le V$ → then W T-invariant if $T(W) \subseteq W$, and $T_W: W \to W$ is restriction of T to W.

- for vector space V and subspace W, QUOTIENT SPACE V/W is a vector space of cosets → $W + v = \{w + v : w \in W\}$.

↳ $\dim V/W = \dim V - \dim W$.

- det of UPPER TRIANGULAR matrix is product of diagonal.

↳ diagonal elements are its evalues.

TRIANGULARISATION THM: every matrix over $\mathbb{C}$ similar to an upper triangular matrix. Generally, for $T: V \to V$ and $c_T(x) = \prod_{1}^{n}(x-\lambda_i)$ THEN $\exists B$ basis of V s.t. $[T]_B$ upper triangular.

CAYLEY-HAMILTON THM: $T: V \to V$ with char poly $p(x)$, $\Rightarrow p(T) = 0$.

EUCLIDEAN ALGORITHM: $f, g \in \mathbb{F}[x]$ with $\deg(f) \geq \deg(g)$

$$\left.\begin{aligned} \Rightarrow f &= q g + r_1 \quad (\deg r_1 < \deg g) \\ g &= q_1 r_1 + r_2 \quad (\deg r_2 < \deg r_1) \\ &\vdots \\ r_{n-1} &= q_n r_n + \underline{r_{n+1}} \quad (\deg r_{n+1} < \deg r_n) \end{aligned}\right\}$$

$r_n = q_{n+1} r_{n+1}$

$\Rightarrow \gcd(f, g) = r_{n+1}$

MINIMAL POLYNOMIAL: $m(x) \in \mathbb{F}[x]$ min poly of $T: V \to V$ if,

$m(T) = 0$; $m(x)$ monic ; $\deg(m)$ is as small as possible s.t. first two hold.

↳ $m_T(x) \mid c_T(x)$ and m_T and c_T have same roots.

PRIMARY DECOMPOSITION THM: $T: V \to V$ with $m_T(x) = \prod_{i=1}^{k} f_i(x)^{n_i}$

$\Rightarrow$ for $V_i = \ker(f_i(T)^{n_i})$, THEN; $V = V_1 \oplus \dots \oplus V_k$

and each V_i T-invariant with each T_{V_i} restriction has min poly $f_i(x)^{n_i}$

↳ $T: V \to V$ diagonalisable iff $m_T(x)$ is product of distinct linear factor.

JORDAN BLOCK: $J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & & 0 & 0 \\ & & \lambda & \dots & 0 & 0 \\ & & & \ddots & \lambda & 1 \\ & & & & 0 & \lambda \end{pmatrix}$ ↗ char poly = min poly $= (x-\lambda)^n$.

JORDAN CANONICAL FORM: A $n \times n$ and char poly is product of linear factors over $\mathbb{F}$. Then A similar to matrix of form;

$$J = J_{n_1}(\lambda_1) \oplus \dots \oplus J_{n_k}(\lambda_k) \text{ where } \sum n_i = n.$$

↳ matrix is uniquely determined by A upto reordering of Jordan blocks.

- in JCF, total dimension of all λ-blocks $= a(\lambda) \leftarrow$ ALGEBRAIC MULT.

  number of λ-blocks $= g(\lambda) \leftarrow$ GEOMETRIC MULT.

  largest power of λ in $m_A(x)$ is size of largest λ-block.

- rank $(A - \lambda I) = n - g(\lambda)$ [RANK-NULLITY].

JORDAN BASIS e.g. $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $c_A(x) = m_A(x) = x(x-1)^3$.

$\Rightarrow$ primary decomp: $V = \ker(A) \oplus \ker(A-I)^3$

$\Rightarrow$ JCF $= J_1(0) \oplus J_3(1)$. For Jordan basis, need basis vector of $\ker A$ AND vector of $\ker(A-I)^3 = v_1$ NOT IN and INDEPENDENT of $\ker(A-I)^2$ and $\ker(A-I)$.

$\Rightarrow$ basis $= \{$basis vector of $\ker A$, $(A-I)^2 v_1$, $(A-I)v_1$, $v_1\}$.

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e.g. (2) $A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$, $c_A(x) = (x-1)^3 = m_A(x)$.

$\hookrightarrow \ker(A-I) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ = evector.

now find $\rightarrow (A-I) \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\leftarrow$ maps to evector i.e. $(A-I)^2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$.

" $\rightarrow (A-I) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\Rightarrow \left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\text{evector}}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Jordan basis

$(A-I)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $(A-I)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

CYCLIC SUBSPACE : $T: V \to V$, then $Z(v, T) = Sp\{v, T(v), T^2(v), \dots\}$.

T-ANNIHILATOR : let $T^k(v)$ be first vector s.t. $T^k = -a_0 v - a_1 T - \dots - a_{k-1} T^{k}$.

$\Rightarrow m_v(x) = x^k + a_{k-1}x^{k-1} + \dots + a_0 \rightarrow$ monic poly of smallest degree s.t. $m_v(T(v)) = 0$.

$$\boxed{\begin{array}{l}\text{CYCLIC DECOMPOSITION THM: } T: V \to V \text{ with } m_T(x) = f(x)^k \\ \text{then } \exists\, v_1, \dots, v_r \in V \text{ s.t. } V = Z(v_1, T) \oplus \dots \oplus Z(v_r, T) \\ \text{each } Z(v_i, T) \text{ has T-annihilator } f(x)^{k_i}.\end{array}}$$

$\Rightarrow \exists$ basis B of V s.t. $[T]_B = C(f(x)^{k_1}) \oplus \dots \oplus C(f(x)^{k_r})$.

RATIONAL CANONICAL FORM: $T: V \to V$ with $m_T(x) = \prod_{i=1}^{k} f_i(x)^{k_i}$

then $\exists$ basis of V, B, s.t. $[T]_B = C(f_1(x)^{k_{11}}) \oplus \dots \oplus C(f_1(x)^{k_{1r_1}}) \oplus \dots$

$$\oplus\, C(f_t(x)^{k_{t1}}) \oplus \dots \oplus C(f_t(x)^{k_{tr_t}})$$

where for each $i \to \underline{k_i = k_{i1}} \geq \dots \geq k_{ir_i}$ [uniquely determined by T]

$\uparrow$ highest power in min poly.

e.g. $c_T(x) = (x^2+x+1)^4(x^3+x+1) \to$ dimension 11

$m_T(x) = (x^2+x+1)^2(x^3+x+1)$ $\boxed{\text{over } \mathbb{F}_2}$

$\text{rank}(A^2+A+I) = 5$

$\hookrightarrow$ RCF of A has: $C((x^2+x+1)^2) \oplus C(x^3+x+1)$

then either $C(x^2+x+1) \oplus C(x^2+x+1)$ <u>OR</u> $C((x^2+x+1)^2)$.

$\text{rank}(A^2+A+I) = 5 \Rightarrow \text{nullity} = 11 - 5 = \underline{6}$

$\therefore$ # of x^2+x+1 blocks $= 6 \div 2 = 3$

$\uparrow$ degree of poly in question.

$\Rightarrow \underline{A \sim C((x^2+x+1)^2) \oplus C(x^2+x+1) \oplus C(x^2+x+1) \oplus C(x^3+x+1)}$.

LINEAR FUNCTIONAL: linear map $\phi: V \to \mathbb{F}$ s.t.

$\underline{\phi(\alpha v_1 + \beta v_2) = \alpha\phi(v_1) + \beta\phi(v_2)}$ $\forall v_i \in V,\ \alpha, \beta \in \mathbb{F}$.

$\hookrightarrow$ e.g. proj map $\pi_i(x_1, \dots, x_n) = x_i$. <u>OR</u> trace map $A \mapsto tr(A)$.

- $\underline{(\phi_1+\phi_2)(v) = \phi_1(v) + \phi_2(v)}$; $\underline{(\lambda\phi)(v) = \lambda \cdot \phi(v)}$.

DUAL SPACE: $V^* = \{\phi \mid \phi: V \to \mathbb{F} \text{ linear funct}^{n}\text{al}\}$ $\to$ set of all linear functionals $\to$ vector space itself.

DUAL BASIS: $B = \{v_1, \dots, v_n\}$ basis of V. Define $\phi_i \in V^*$ by:

$\underline{\phi_i(\sum \alpha_j v_j) = \alpha_i}$ (OR $\phi_i(v_j) = \delta_{ij}$)

$\Rightarrow \{\phi_1, \dots, \phi_n\}$ basis of V^*.

$\hookrightarrow$ e.g. $V = \mathbb{F}^n$ with standard basis $e_1, \dots, e_n$

dual basis is $\pi_1, \dots, \pi_n$ (proj maps).

ANNIHILATOR : for $X \subseteq V$, X° of X is $X^\circ = \{\phi \in V^* : \phi(x) = 0 \ \forall x \in X\}$

↳ NIB: X° subspace of V^*.

$\boxed{\text{for } W \leqslant V, \quad \dim W^\circ = \dim V - \dim W.}$

INNER PRODUCT : map $V \times V \to \mathbb{F}$ by $(u, v) \in \mathbb{F}$ s.t.

* ① $(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 (v_1, w) + \lambda_2 (v_2, w)$ left-linear

② $(v, w) = \overline{(w, v)}$ ③ $(v, v) > 0$ for $v \neq 0$.

• if $(v, w) = (v, x)$ $\forall v \in V \Rightarrow w = x$.

↳ e.g. DOT PRODUCT $x \cdot y = \sum x_i \bar{y}_i = x^T \bar{y}$.

(or $A = \bar{A}^T$)

• Matrix of an inner product is HERMITIAN → $A^T = \bar{A}$.

↳ e.g. $(v, w) = [v]_B^T A [\bar{w}]_B$ for basis B.

Unitary is $A^T \bar{A} = I$

POSITIVE-DEFINITE: if Hermitian A s.t. $x^T A \bar{x} > 0$.

(↳ makes sense only for Hermitian since they have non-complex evalues. "positive" doesn't make sense in complex)

CAUCHY-SCHWARZ : $|(u, v)| \leqslant \|u\| \cdot \|v\|$.

GRAM-SCHMIDT : create orthonormal basis, given any basis.

- start with $v_1, \ldots, v_n$ basis of V. Create orthonormal $u_1, \ldots, u_n$.

$u_1 = \dfrac{v_1}{\|v_1\|}$, $w_2 = v_2 - (v_2, u_1) u_1$ so $u_2 = \dfrac{w_2}{\|w_2\|}$.

$w_3 = v_3 - (v_3, u_1) u_1 - (v_3, u_2) u_2$, so $u_3 = \dfrac{w_3}{\|w_3\|}$ etc.

generally, $w_i = v_i - (v_i, u_1) u_1 - \ldots - (v_i, u_{i-1}) u_{i-1}$ → then normalise.

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for $W \subseteq V$, $W^\perp = \{u \in V : (u, w) = 0 \ \forall w \in W\}$ set of vectors in V which are orthogonal to everything in W.

↳ $W^\perp$ subspace of V,

$\boxed{\bullet \text{ for } W \leqslant V, \quad V = W \oplus W^\perp}$

for linear map $T: V \to V$, $\exists\, T^*: V \to V$ s.t. $(T(u), v) = (u, T^*(v))$

T^* is ADJOINT of T and if $T = T^*$ then SELF ADJOINT.

Given orthonormal basis $E = \{v_1, \dots, v_n\}$ and $T: V \to V$

THEN, $[T^*]_E = \overline{[T]_E}^T$.

→ if $T = T^*$ then T hermitian.

SPECTRAL THM: Given self-adjoint $T: V \to V$, THEN V has orthonormal set of T-eigenvectors.

Given symmetric / Hermitian A, then $\exists$ orthogonal / unitary P s.t. $P^{-1}AP$ diagonal.

BILINEAR FORM: map: $V \times V \to \mathbb{F}$ both left- and right-linear.

↳ e.g. $(A, B) = \text{tr}(AB)$ OR $V = \mathbb{F}^n$, $(u, v) = u^T A v$ ← all bilinear forms like this

SYMMETRIC if $(u, v) = (v, u)$; SKEW-SYMMETRIC if $(u, v) = -(v, u)$ $\forall u, v \in V$

↳ $A^T = A$. ↳ $A^T = -A$.

(again, for $W \subseteq V$, $W^\perp = \{v \in V : (v, w) = 0\ \forall w \in W\}$. → subspace of V).

NON-DEGENERATE: if $V^\perp = 0$ i.e. $(u, v) = 0\ \forall v \in V \Rightarrow u = 0$.

↳ only vector 'perpendicular' to all others in space is $\underline{0}$.

↳ for non-degen $\dim W^\perp = \dim V - \dim W$

analogous to W^0. $W^0 = \{f_v \in V^* : f_v(w) = 0\ \forall w \in W\}$

MATRIX of a BILINEAR FORM non-degen iff matrix invertible.

CONGRUENT: A, B congruent if $\exists P$ s.t. $B = P^T A P$.

then if $(u, v)_1 = u^T A v$ AND $(u, v)_2 = u^T B v$

↳ $(\,,\,)_1$ and $(\,,\,)_2$ EQUIVALENT.

any non-degen skew-symmetric equivalent to $(x, y) = x^T J_m y$

for $J_m = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $n = 2m$.

- any symmetric bilinear form equivalent to diagonal bilinear form (for symm A, $A = PDP^T$) $\rightarrow$ SPECTRAL.

ALGORITHM to compute ORTHOGONAL BASIS given BILINEAR FORM:

$\hookrightarrow$ e.g. $(x, y) = x_1 y_2 + x_2 y_1 = (x_1 \; x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

pick $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $(v_1, v_1) = 2 \neq 0$ ✓. Now want $v_2 \in v_1^{\perp}$ s.t. $(v_2, v_2) \neq 0$. clearly, $v_1^{\perp} = sp\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓.

($\hookrightarrow$ if more dimensions, find $v_3 \in \{v_1, v_2\}^{\perp}$ s.t. $(v_3, v_3) \neq 0$ etc.)

QUADRATIC FORM: $Q: V \rightarrow \mathbb{F}$, $Q(v) = (v, v)$ s.t. $(,)$ is symmetric bilinear form on V.

change of variables $\rightarrow$ $x = Py$, then;

$$Q(x) = (Py)^T A (Py) = y^T (P^T A P) y = Q'(y)$$

$\hookrightarrow$ Q and Q' EQUIVALENT since matrices CONGRUENT.

$\nearrow P = (v_1 | v_2)$ for orthogonal basis v_1, v_2 then $x = Py$ admits diagonal $Q(x)$

every quadratic form can be diagonalised:

$$Q_D(x) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 = x^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} x.$$

For non-degen quadratic form $Q: V \rightarrow \mathbb{F}$,

- if $\mathbb{F} = \mathbb{C}$ then every Q equivalent to one with I_n

$$Q_0(x) = x_1^2 + \dots + x_n^2 = x^T I_n x$$

- if $\mathbb{F} = \mathbb{R}$ then every Q equivalent to one with I_{pq}

$$Q_{pq}(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 = x^T I_{pq} x$$

where $I_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

(• if $F = \mathbb{Q}$, then infinitely many non-equivalent Q.)